Group theoretic reduction of Laplacian dynamical problems on fractal lattices

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Discrete forms of the Schrödinger equation, the diffusion equation, the linearized Landau-Ginzburg equation, and discrete models for vibrations and spin dynamics belong to a class of Laplacian-based finite difference models. Real-space renormalization of such models on finitely ramified regular fractals is known to give exact recursion relations. It is shown that these recursions commute with Lie groups representing continuous symmetries of the discrete models. Each such symmetry reduces the order of the renormalization recursions by one, resulting in a system of recursions with one fewer variable. Group trajectories are obtained from inverse images of fixed and invariant sets of the recursions. A subset of the Laplacian finite difference models can be mapped by change of boundary conditions and time dependence to a diffusion problem with closed boundaries. In such cases conservation of mass simplifies the group flow and obtaining the groups becomes easier. To illustrate this, the renormalization recursions for Green functions on four standard examples are decoupled. The examples are (1) the linear chain, (2) an anisotropic version of Dhar's 3-simplex, similar to a model dealt with by Hood and Southern, (3) the fourfold coordinated Sierpiński lattice of Rammal and of Domany *et al.*, and (4) a form of the Vicsek lattice. Prospects for applying the group theoretic method to more general dynamical systems are discussed. [S1063-651X(97)11506-9]

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I. INTRODUCTION

The purpose of this paper is to demonstrate a group theoretic reduction of dynamical systems arising from real-space renormalization of Laplacian problems on regular fractal lattices. Thus the wave mechanical or diffusion Green functions for a variety of lattice types can be analyzed rather completely. We define first the physical models and then the renormalization procedure. Finally we present the method of reduction with several examples.

The diffusion equation, the Schrödinger equation, and the classical wave equation are each based on Laplacian operators. They relate to one another through changes in time dependence and of boundary conditions. In simple geometries one can apply Lie theory to construct solutions of general partial differential equations (PDEs) by making use of continuous symmetries. However, since these must be symmetries of the boundary conditions as well as the differential equations, the method is often less useful for regions with very complicated boundaries.

A standard way to treat Laplacian-based PDEs in complex spatial regions is to make the space coordinates discrete. In general terms, space is replaced by a graph serving as a quadrature grid and the PDEs are replaced by a system of coupled ordinary differential equations, one for each node, so that time remains continuous but space becomes discrete. Thus the original continuum problem is embedded in a class that also includes discrete lattice models that do not correspond to a unique continuum limit. We will take the discrete models to be more fundamental.

To generalize the Laplacian operator on a graph we take the diffusion equation as a guide [1]. A vertex current rule similar to Kirchhoff's law is adopted to ensure conservation of material. To relate bond currents to the node concentrations we adopt a generalized Fick's law. In this way one constructs finite difference diffusion schemes based only on adjacency rather than geometrical distances or angles. By comparing with the diffusion equation, these bond and vertex rules define difference Laplacians that can be transported to other physical problems, such as wave propagation.

Once the problem is replaced by a system of differential equations it can be reduced to linear algebra by Laplace or Fourier transform. Finding symmetry transformations is more difficult for the resulting difference equations, but alternative solution methods become available, such as direct numerical solution, graph theoretic methods, or real-space renormalization. In the latter method one finds recursion relations (usually approximate) for some set of properties on one length scale in terms of the same properties on another length scale.

The fractal paradigm [2] is a natural one for classifying scaling laws in structures where there is some form of self-similarity. General scaling theory of diffusion in fractal structures is well developed. It is reviewed by Havlin and Ben-Avraham [3]. It is sometimes argued [4–6] that regular fractal lattices capture important aspects of critical percolation clusters, aerogels, or even amorphous solids while avoiding the difficulty of true randomness. However, another justification for studying regular fractals, in our view, is that one can obtain many analytical results. Often problems not

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solvable on Euclidean lattices become solvable on regular fractals. Exact solutions give insight different from that afforded by the approximate solution of more realistic models.

The structures dealt with here are hierarchical graphs such as the ones first introduced by Dhar [7]. Scaling properties, especially exponents, are known for a variety of regular fractal lattices [7,4]. The Laplacian-based problems, including the Schrödinger equation, the scalar wave equation, and the diffusion equation are the most relevant to the discussion below. Domany *et al.* and Rammal [8,9] renormalized the Schrödinger eigenstates (or vibrational modes) of a Sierpiński lattice to obtain the energies (frequencies) and the wave amplitudes. The spectrum consists of a Cantor set of eigenvalues corresponding to hierarchical wave functions, and isolated frequencies in the gaps of the Cantor set that correspond to molecular localized modes. The latter are symmetry decoupled with wave amplitudes that vanish outside a circle of fixed radius. These spectral properties are considered characteristic of finitely ramified fractals.

Rammal [9] and Tremblay and Southern [10] renormalized the Green function generating function by decimation in order to study integral spectral properties of fractals. Alexander [11] introduced the transfer matrix renormalization on regular fractals. He demonstrated that a substantial simplification results from reduction of the Green functions with respect to the point symmetries of the lattice. We review below certain aspects of Alexander's synthetic method in which Green functions are expressed in a symmetry adapted basis projected from irreducible representations of the point group. The transfer matrix method applied only to a small set of pivotal Green functions has been used by several authors [12,13]. For the pivotal set one obtains recursion relations in the form of rational functions, which can be considered as a discrete dynamical system or as coupled difference equations. Application of the method to transport problems involving the diffusion equation has been presented in detail elsewhere [14]. Below we show how to reduce the order of these dynamical systems. In some cases the recursions decouple completely, resulting in explicit solutions.

The Lie theory of integration of differential equations subsumes all other integration methods. Lie showed that all known integration methods are equivalent to finding continuous groups that permute the solutions and then using these groups to reduce the order [15–17]. For equations of second order or higher the group theoretic method also gives an algorithm for finding the groups. One problem with applying Lie theory to difference equations is that there is no systematic way to find a continuous symmetry, although once a group is found that commutes with the recursions it leads to a reduction of order, just as in the case of differential equations.

The group theoretic reduction of systems of differential equations makes use of infinitesimal group generators. These are differential operators that when exponentiated reproduce the action of the group on functions of the basic variables. An important tool is the prolongation of a generator, which reproduces the action of the group on functions that include also derivatives. In the approach taken by Maeda [18] and by Quispel and Sahadevan [19] one extends prolongations to include finite differences and obtains a functional equation for the group generators. While this approach is useful in principle, it is less useful than the one developed for integrating differential equations. We elect not to follow it, although material given below on reduction of order produced by a continuous symmetry is taken mostly from Ref. [19]. It is outlined here for completeness.

In the following sections we apply real-space renormalization to find Green functions for Laplacian-based equations in difference form on several regular fractal lattices. We show how the method of Lie can be used to reduce the order of the recursion relations. In some cases this results in a solution for Green functions at specific values of the energy parameter as functions of the lattice size.

II. LAPLACIAN PROBLEMS ON LATTICES

In this section we review the standard difference models for the diffusion equation, the scalar wave equation, and the Schrödinger equation while noting the relationships among them, particularly with regard to changes in boundary conditions.

The recipe given above leads to a typical diffusion model of the form

$$\frac{d}{dt}c_i(t)=\sum_j k_{ij}[c_j(t)-c_i(t)]+f_i(t), \tag{2.1}$$

where k_{ij} is the microdiffusivity or bond conductance between sites i and j. The source $f_i(t)$ is added as a formal convenience. We can reorganize the sum using

$$K_{ij}=k_{ij}-\delta_{ij}\sum_n k_{in}, \tag{2.2}$$

so that

$$\frac{d}{dt}c_i(t)=\sum_j K_{ij}c_j(t)+f_i(t). \tag{2.3}$$

The summation on the right side is the discrete analog of an integral operator with kernel K. Boundary conditions are included, as long as the index range covers the entire graph (closed system). In vector form,

$$\frac{d}{dt}c(t)=Kc(t)+f(t). \tag{2.4}$$

Laplace transforming gives a linear algebraic equation

$$s\hat{c}(s)-c(0)=K\hat{c}(s)+\hat{f}(s), \tag{2.5}$$

or

$$\hat{c}(s)=G^{(d)}(s)[\hat{f}(s)+c(0)], \tag{2.6}$$

where entry $G_{ij}^{(d)}(s)$ of the resolvent

$$G^{(d)}(s)=(s-K)^{-1}, \tag{2.7}$$

defined for s outside the spectrum of the matrix K, is a Green function for diffusion from j to i with mass conservation.

Sometimes it will be convenient to change K by changing only a few bonds. This happens when introducing external connections, deriving renormalization recursions, or trans-

forming between problems. Thus we find the following standard result useful. Suppose $G_A=(z-A)^{-1}$ and $G_C=(z-C)^{-1}$ for general complex z where $C=A+B$. Simple manipulations yield

$$G_C=G_A+G_ABG_C. \tag{2.8}$$

This relation is particularly useful when B is sparse.

A difference model for the scalar wave equation, often applied to vibrations, differs from Eq. (2.1) in two ways. The time dependence and boundary conditions are different. When there are no external springs, the typical scalar vibration model is exactly as in Eq. (2.1), except that the time derivative is second order. Local displacements $u_i(t)$ replace the concentrations, k_{ij} becomes a spring constant, and $f_i(t)$ is an external driving force. Thus

$$\frac{d^2}{dt^2}u_i(t)=\sum_j k_{ij}[c_j(t)-c_i(t)]+f_i(t). \tag{2.9}$$

Since there are no external springs, no tension is maintained and the structure has many zero-energy distortions or zero-frequency modes. To supply external springs for anchors one can use a sparse matrix B that is zero except at the physical boundary of the vibrating structure. Thus for the anchored model

$$K_{ij}=k_{ij}-\delta_{ij}\left(\sum_n k_{in}+\sum_b{}' k_{ib}\right), \tag{2.10}$$

with the primed sum running over external springs attached to site i. Whence in order to obtain this model from Eq. (2.2),

$$B_{ij}=-\delta_{ij}\sum_b{}' k_{ib}. \tag{2.11}$$

It is often useful to include a viscous damping term. For the wave equation a Fourier transform is more natural, so the analog of vector Eq. (2.5) for the damped, anchored vibration model is

$$-\omega^2\hat{u}(\omega)-2\frac{i\omega}{\tau}\hat{u}(\omega)=(K+B)\hat{u}(\omega)+\hat{f}(\omega). \tag{2.12}$$

The damping time τ is assumed the same throughout for simplicity. The corresponding Green functions are entries of

$$G^{(v)}(\omega)=-\left(\omega^2+\frac{2i\omega}{\tau}+K+B\right)^{-1}. \tag{2.13}$$

A difference model for the one-electron Schrödinger equation (no magnetic field) is

$$i\frac{d}{dt}\psi_i(t)=\sum_j H_{ij}\psi_j(t). \tag{2.14}$$

The difference Hamiltonian H is typically a Laplacian like K defined in Eq. (2.10) with $k_{ij}=A_{ij}$, which is 1 if i and j are connected on the graph and 0 otherwise. Or else when the wave function is expanded in a linear combination of localized basis functions, H can be a more general symmetric matrix. We put Eq. (2.14) in vector form and Fourier transform from time t to the complex energy $E+i\eta$. The Green functions are then entries of

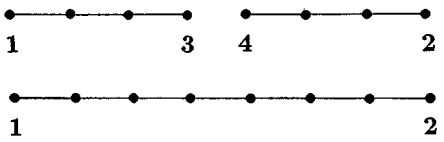

FIG. 1. Recursive construction of linear chain. Two copies of generation n are connected to form generation $n+1$. Numbers index connection sites.

$$G^{(s)}(E+i\eta)=(E+i\eta-H)^{-1}. \tag{2.15}$$

Thus each of the common Laplacian difference models leads to Green functions that are entries of

$$G(z)=(z-H)^{-1}, \tag{2.16}$$

where the complex parameter z signifies time dependence. For diffusion $z=s$, for the wave equation $z=-\omega^2-2i\omega/\tau$, and for the Schrödinger equation $z=E+i\eta$. The matrix H is the Hamiltonian for the Schrödinger equation, K for diffusion, and $K+B$ for the wave equation. In any case, boundary conditions can be adjusted using a boundary matrix B. Thus it is possible to map back and forth between problems.

III. RENORMALIZATION AND POINT SYMMETRY REDUCTION ON REGULAR FRACTALS

A trivial example is a linear chain of 2^n sites as shown in Fig. 1. Imagine constructing the chain in steps or generations indexed by n. At generation n the lattice consists of 2^n sites. In the next step, two generation n chains are joined by a bond to form a chain of length 2^{n+1}, which makes generation $n+1$.

Consider a Schrödinger problem on the linear chain. For the Hamiltonian H we take the adjacency matrix $H=A$. [This is related to $K+B$ of Eq. (2.12) for the vibration problem when all spring constants are unity, $k_{ij}=1$, by $H=K+B+2$. Alternatively one could use $H=2-A$ so that the spectrum would extend from 0 to positive energy.] The Green functions of interest are entries of $G(z)=(z-A)^{-1}$.

The renormalization method is discussed in detail in Refs. [12,13,6,14]. We illustrate it for the case of the linear chain. The basic tool is Eq. (8). Consider the process of constructing generation $n+1$. Let G_A be the set of Green functions on the two disconnected pieces of generation n, and let B provide the bond to connect them. Thus, referring to Fig. 1, $B_{ij}=\delta_{i3}\delta_{j4}+\delta_{i4}\delta_{j3}$, so B has only two nonzero entries. The pivotal Green functions are $x=G_{A11}(z)$, $y=G_{A31}(z)$ on generation n and $X=G_{C11}(z)$, $Y=G_{C21}(z)$ on generation $n+1$.

From Eq. (2.8),

$$X=x+yG_{C41}(z),\quad Y=yG_{C31}(z),$$
$$G_{C31}(z)=y+xG_{C41}(z),\quad G_{C41}(z)=xG_{C31}(z). \tag{3.1}$$

These yield the recursions

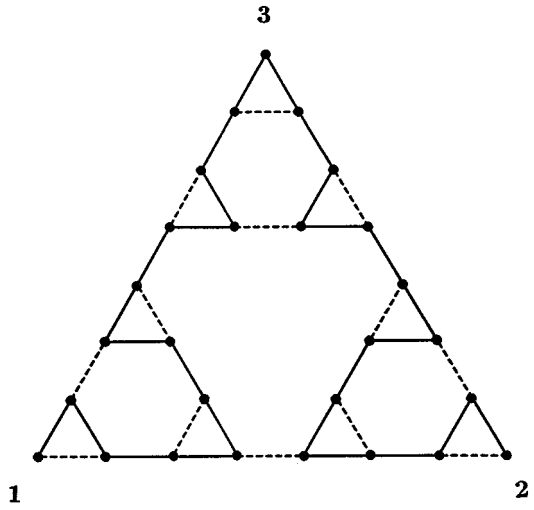

FIG. 2. Anisotropic 3-simplex. Sites 1, 2, and 3 are connection points.

$$X=x+\frac{xy^2}{1-x^2},\quad Y=\frac{y^2}{1-x^2}. \tag{3.2}$$

For any lattice constructed by attaching generation n blocks together to make generation $n+1$, one obtains similar recursions for a set of pivotal Green functions.

Alexander [11] showed that a considerable simplification results from transforming to a point symmetry adapted basis. We will not use all of his machinery, but we will make use of the basic idea. Let $\hat{e}_1$ and $\hat{e}_2$ be basis vectors with components $\hat{e}_{1i}=\delta_{i1}$, $\hat{e}_{2i}=\delta_{i2}$ in the space of the vector ψ of Eq. (2.14). The point group of the structure has two operations, the identity and reflection. The two irreducible representations are even (character 1) and odd (character -1). A symmetry adapted basis is prepared using the van Vleck machine [20]. In this simple case

$$\hat{e}_{\text{even}}=\frac{1}{\sqrt{2}}(\hat{e}_1+\hat{e}_2),\quad \hat{e}_{\text{odd}}=\frac{1}{\sqrt{2}}(\hat{e}_1-\hat{e}_2). \tag{3.3}$$

The point symmetrized Green functions are

$$p=\hat{e}_{\text{even}}\cdot G_C\hat{e}_{\text{even}}=x+y,$$
$$q=\hat{e}_{\text{odd}}\cdot G_C\hat{e}_{\text{odd}}=x-y. \tag{3.4}$$

Using these coordinates, the recursions become

$$P=p-\frac{(1-p)(p-q)}{2-p-q},\quad Q=q+\frac{(p-q)(1+q)}{2+p+q}. \tag{3.5}$$

Evidently the lines $p=1$ and $q=-1$ are invariant under the mapping defined by Eq. (3.2). These lines are related to mass conservation in the diffusion problem, as we shall see. The line $q=p$ is fixed because in the limit of a very long chain there is no propagation from one end to the other, either by classical diffusion or electron quantum mechanics, so iteration of Eq. (3.2) leads to $y=0$ for almost all energies.

Figure 2 shows a lattice model for anisotropic diffusion on a 3-simplex lattice. Dashed lines correspond to small diffusivity $k_{ij}=k$ and solid lines correspond to $k_{ij}=1$. Hood and Southern studied spectral properties on a model similar to this [21]. Recently Adrover *et al.* have studied scaling crossover for diffusion in this model [22]. We consider the Schrödinger equation for definiteness with H the generalized adjacency matrix, A_{ij} being 1 for strong bonds, k for weak bonds, and zero otherwise. The lower corner sites are 1 and 2 and the upper corner is 3. Due to anisotropy, four pivots are necessary, $x=G_{11}(z)$, $y=G_{21}(z)$, $u=G_{33}(z)$, and $v=G_{31}(z)$. The recursions are obtained as in the case of the chain by considering the connection of three blocks together via two strong and one weak bond. However, the recursions obtained in this way are a bit complicated and contain the parameter k. To simplify them and scale k out, we make use of the symmetrized coordinates

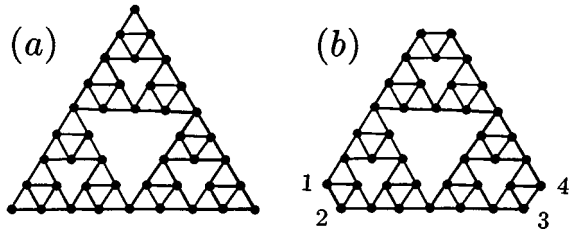

FIG. 3. Fourfold coordinated Sierpiński lattice showing removal of corner sites to facilitate writing recursions.

$$p=x+y,\quad q=x-y,\quad r=uk,\quad s=v\sqrt{k}. \tag{3.6}$$

Then the recursions become

$$P=q+\frac{(1-pq)(p-q-pr+qr+2s^2)}{2-p^2-pq-2r+p^2r+pqr-2ps^2},$$
$$Q=q+\frac{(1+q)(1-q)(p-q+pr-qr-2s^2)}{2-pq+q^2+2r-pqr-q^2+2qs^2},$$
$$R=r+\frac{2s^2(p+q-pr-qr+2s^2)}{2-p^2-pq-2r+p^2r+pqr-2ps^2}, \tag{3.7}$$
$$S=\frac{s(p-q-pr+qr+2s^2)}{2-p^2-pq-2r+p^2r+pqr-2ps^2}.$$

Again they comprise a mapping defined by a set of rational functions, i.e., ratios of polynomials, and certain fixed manifolds are evident.

Figure 3 illustrates construction of the fourfold coordinated Sierpiński lattice of Refs. [8] and [9]. Consider diffusion. To apply the same kind of renormalization directly in this case requires removing the corner sites as shown in Fig. 3(b). Thus there are six distinct connection points and five independent Green functions per generation. With corner vertices as indicated in the figure, a pivotal set is $x=G_{11}(z)$, $y=G_{32}(z)$, $t=G_{21}(z)$, $u=G_{41}(z)$, and $v=G_{31}(z)$. A more convenient set consists of the symmetry adapted variables [11]

$$p=x+t,\quad q=x-t,\quad r=u+v,$$
$$s=u-v,\quad m=u+2v+y. \tag{3.8}$$

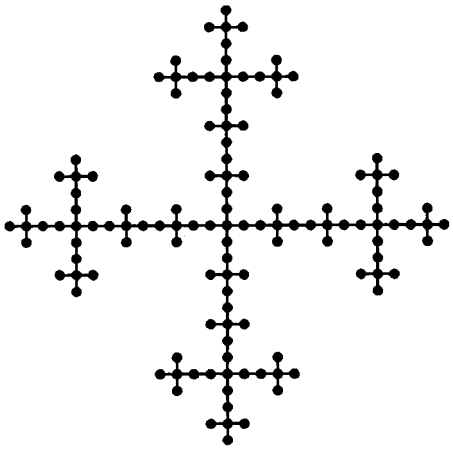

FIG. 4. Vicsek lattice.

The recursions are

$$P=p+\frac{m^2(1-4p)}{(1-2m-4p)(1+m-4p)},$$

$$Q=q+\frac{(m-2r)^2}{1+m-4p},$$

$$R=\frac{m(2m^2+r-2mr-4pr)}{(1-2m-4p)(1+m-4p)}, \tag{3.9}$$

$$S=-\frac{(m-2r)r}{1+m-4p},$$

$$M=\frac{m^2(1+2m-4p)}{(1-2m-4p)(1+m-4p)}.$$

In this case, since the corners have been removed, the pivotal Green functions are not the ones for the closed system. Rather they are the ones appropriate for attaching a reservoir at each connection site.

Figure 4 shows a Vicsek lattice [23]. Dynamical problems on this lattice have been treated by several authors recently in connection with modes of a fractal drum [24,25]. Diffusion of material and of vibrational energy on this lattice are discussed elsewhere [26]. We first consider diffusion. Pivotal Green functions are $x=G_{11}(z)$ and $y=G_{21}(z)$, where 1 and 2 are two distal corners. The recursions in (x,y) become [26]

$$X=x-\frac{y^2(1+2x+2y)}{(1+2x-y)(1+2x+3y)},$$

$$Y=\frac{y^3}{(1+2x-y)(1+2x+3y)}. \tag{3.10}$$

Since the Vicsek lattice is not a rigid structure, the point symmetry reduction is less profitable in this case. As usual the recursions are rational, and the line $y=0$ is fixed.

The examples above are typical renormalization recursion relations for Laplacian dynamical problems. Each one can be iterated to obtain Green functions at any desired generation. It is useful to think of the recursion formulas as mappings of the pivot space into itself, thus defining a discrete dynamical system. In this sense they are systems of nonlinear difference equations. A solution to the difference equations in each case would be formulas giving the Green functions at generation n as explicit functions of n. As we shall see below, we are able to obtain an infinite number of such solutions in many cases. These exact solutions are obtained at a certain Cantor set of initial conditions (Cantor set in z). In the ideal situation one could find formulas for the pivots as functions of both n and z for any z. We will not be able to do this, in general, because the Cantor set of exactly solvable z values is determined by the chaotic dynamics of an irreducible set of maps. In the case of the linear chain, the irreducible set contains only the solvable (Tchebyshev) case of the quadratic polynomial map. Therefore the map is integrable and the problem is completely solvable.

IV. LIE GROUPS AND REDUCTION OF ORDER

Tools presented in this section are used in the following one to decouple recursion equations. First we show what it means for a dynamical system to admit a continuous group. We show that when the system is expressed in terms of the canonical variables of the group its order is reduced. In the following section we will consider strategies for finding such a group.

Consider a discrete dynamical system $\mathcal{S}$ of the generic form

$$X=f(x,y,z), \quad Y=g(x,y,z), \quad Z=h(x,y,z), \tag{4.1}$$

with $x=x_n$ and $X=x_{n+1}$, etc., as usual. These recursions are also autonomous difference equations, meaning that n does not appear explicitly. We consider the action of a continuous, one-parameter group $\mathcal{G}$ of transformations acting on the variables. Thus (x,y,z) are replaced by $(\tilde{x},\tilde{y},\tilde{z})$ such that

$$\tilde{x}=u(x,y,z,\alpha), \quad \tilde{y}=v(x,y,z,\alpha), \quad \tilde{z}=w(x,y,z,\alpha). \tag{4.2}$$

The substitutions form a group with respect to composition. For the group operation the parameters add so that $\alpha=0$ is the identity and $-\alpha$ indexes the inverse element. The functions $u(x,y,z,\alpha)$, etc. are continuous in α. Quispel and Sahadevan [19] point out that although the system $\mathcal{S}$ is autonomous, the group transformations $\mathcal{G}$ may depend on n in the most general case. However, for the examples below we need not include the n dependence. Hence the discussion simplifies.

To say the system $\mathcal{S}$ admits the group $\mathcal{G}$ is to say that it does not matter whether one makes the substitution Eq. (4.2) before or after applying the recursions Eq. (4.1), the result is the same. In other words $u(X,Y,Z,\alpha)=f(\tilde{x},\tilde{y},\tilde{z})$, etc., or, in full detail,

$$
\begin{aligned}
&u(f(x,y,z),g(x,y,z),h(x,y,z),\alpha)\\
&\quad = f(u(x,y,z,\alpha),v(x,y,z,\alpha),w(x,y,z,\alpha)),\\
&v(f(x,y,z),g(x,y,z),h(x,y,z),\alpha)\\
&\quad = g(u(x,y,z,\alpha),v(x,y,z,\alpha),w(x,y,z,\alpha)),\\
&w(f(x,y,z),g(x,y,z),h(x,y,z),\alpha)\\
&\quad = h(u(x,y,z,\alpha),v(x,y,z,\alpha),w(x,y,z,\alpha)).
\end{aligned}
\tag{4.3}
$$

If we think of both the group elements of $\mathcal{G}$ defined in Eq. (4.1) and the recursions $\mathcal{S}$ of mappings, we may say equivalently that $\mathcal{G}$ commutes with $\mathcal{S}$ or that $\mathcal{S}$ is invariant under $\mathcal{G}$. The meaning is that each group element g_α takes each solution of the difference equations into some solution (possibly the same).

Interest in finding a group admitted by a given system stems from the fact that the group can be used to reduce the order of the system. The procedure is first to find canonical coordinates of $\mathcal{G}$. When $\mathcal{S}$ is transformed into canonical coordinates, one of the recursions equations decouples. By canonical variables of the group we mean a set (a,b,c) in terms of which the action analogous to Eq. (4.2) becomes

$$
\tilde{a}=a, \quad \tilde{b}=b, \quad \tilde{c}=c+\alpha. \tag{4.4}
$$

Making use of this simplification in Eq. (4.3) one has

$$
\begin{aligned}
&A=F(a,b,c+\alpha), \quad B=G(a,b,c+\alpha),\\
&C+\alpha=H(a,b,c+\alpha),
\end{aligned}
\tag{4.5}
$$

where $F(a,b,c)$, etc., are defined by changing coordinates in Eq. (4.1). Since Eq. (4.5) should hold for all α, one can set $\alpha=-c$, so in fact the recursions for a and b do not depend on c and the recursion for c is decoupled:

$$
A=F(a,b,0), \quad B=G(a,b,0), \quad C=c+H(a,b,0). \tag{4.6}
$$

The solution for c is just a summation of a function of the solutions for a and b.

The method of finding canonical variables in this case is straightforward and can be found in Ref. [16]. In essence it differentiates $(\tilde{a},\tilde{b},\tilde{c})$ with respect to α in two different ways, one explicitly using Eq. (4.4) and one via the chain rule from $\tilde{a}=\tilde{a}(\tilde{x},\tilde{y},\tilde{z})$, etc. Setting $\alpha=0$ gives a system of three first order linear PDEs for a, b, and c. In many cases they can be solved quite easily.

The reader can see that the difficult part is to find the group. We know of no systematic way for general dynamical systems. However, for systems arising from renormalization of Laplacian lattice problems the strategies we present in the next section have been found quite successful.

V. FINDING GROUPS

The method depends on the relationship between invariant sets of $\mathcal{S}$ and group trajectories of $\mathcal{G}$. We begin with informal comments on invariant sets, then using the examples we show how groups can be found from circuit theory, and how to use them to reduce the order of the recursions. In this way electric circuit theory integrates the Schrödinger problem on a lattice.

A substitution group $\mathcal{G}$ of the form Eq. (4.2) defines a set of mappings of the pivot space into itself, one mapping for each α value. Let g_α be the group element corresponding to the particular value α. In this sense $\mathcal{G}$ is on an equal footing with the dynamical system $\mathcal{S}$, Eq. (4.1). The two important differences are that each transformation of $\mathcal{G}$ is one to one, and the image $(\tilde{x},\tilde{y})$ of a given point (x,y) is a continuous function of α, while of course neither of these two properties needs be true for the action of $\mathcal{S}$. We will use the following observations.

Let P be a fixed point of $\mathcal{S}$. Suppose g_α takes P into $P(\alpha)$. Since $\mathcal{G}$ commutes with $\mathcal{S}$, $P(\alpha)$ is also a fixed point of $\mathcal{S}$. If there is a neighborhood of P that contains no other fixed point of $\mathcal{S}$, then P must also be a fixed point of $\mathcal{G}$, since $P(\alpha)$ must tend to P continuously as $\alpha\to 0$. The only way this can happen if P is isolated is that $P(\alpha)$ is actually the same as P for all α near 0. Thus an isolated fixed point of the system must also be a fixed point of the group.

An invariant set of $\mathcal{S}$ is a set T, which $\mathcal{S}$ takes into itself. In other words, each point in T maps by recursions into another point in T. It follows that if the curve $P(\lambda)$ is invariant under the action of $\mathcal{S}$, where λ parametrizes the curve, then each element of $\mathcal{G}$ must take $P(\lambda)$ into a fixed point or some invariant curve. This may be either the same or another invariant curve. If it is the same one for all α, then the curve is also a group trajectory. Thus in particular a fixed line of the recursion map must be a group trajectory. If $P(\lambda)$ does not go into the same curve, then $\mathcal{S}$ must have a higher dimensional invariant set $P(\alpha,\lambda)$, where $P(\alpha,\lambda)$ is the image under g_α of $P(\lambda)$. In the latter case, the group permutes invariant curves within the subset $P(\alpha,\lambda)$ of T.

Next suppose point Q is in the preimage of an isolated fixed point P of $\mathcal{S}$. Since $\mathcal{S}$ admits $\mathcal{G}$, the group takes Q into $Q(\alpha)$, which must be in the preimage with respect to $\mathcal{S}$ of $P(\alpha)$. Thus a one-dimensional preimage of an isolated fixed point of the system comprises a set of group trajectories connected together end to end by group fixed points. In the simplest case, it is a single trajectory with fixed points at infinity.

Finally, suppose $P(\alpha)$ is a group trajectory, and suppose $\mathcal{S}$ takes $P(\alpha)$ into another curve $Q(\alpha)$. Because the group commutes with the dynamical system, $Q(\alpha)$ is also a group trajectory. We conclude that the recursion relations map the trajectories of the group into one another.

The existence of curves permuted by the renormalization recursion relations has been reported for a number of lattice problems [27–29]. To discover an invariant family of curves, and hence perhaps a group, one can concentrate first on the fixed and invariant manifolds of the renormalization map $\mathcal{S}$. It turns out that the connection to the renormalization of a resistor network becomes important. A detailed treatment of fractal resistor networks is given by Adler [30].

Consider the recursion Eqs. (3.5) for Schrödinger Green functions (x,y) for the end points of a linear chain. We demonstrate the reduction process for this simple example first. The more difficult lattice examples presented below follow the same general pattern.

The connection to diffusion is straightforward. Begin with

diffusion in a completely closed chain with K as in Eq. (2.2). Now attach the ends of the chain at sites 1 and N to reservoirs of concentration c_a and c_b, respectively, using bonds of diffusivity 1. The equation is

$$\frac{d}{dt}c_i(t)=\sum_j K_{ij}c_j(t)+\delta_{i1}(c_a-c_1(t))+\delta_{iN}(c_b-c_N(t)). \tag{5.1}$$

Using $H=K+B$ with $B_{ij}=-\delta_{i1}\delta_{j1}-\delta_{iN}\delta_{jN}$ gives

$$\frac{d}{dt}c_i(t)=\sum_j H_{ij}c_j(t)+\delta_{i1}c_a+\delta_{iN}c_b, \tag{5.2}$$

where H is the same as in the Schrödinger problem leading to Green functions recursions, Eqs. (3.2). The remaining reservoir terms act as sources. Thus one finds the following for the end-point concentrations in the Laplace domain:

$$\hat{c}_1(s)=x(s)c_a/s+y(s)c_b/s,$$
$$\hat{c}_2(s)=y(s)c_a/s+x(s)c_b/s. \tag{5.3}$$

From the circuit analogy, the electrical resistance from end to end should be the steady-state limit ($t\to\infty$) of the ratio of $\Delta c(t)/I(t)=[c_1(t)-c_2(t)]/[c_a-c_1(t)]$, where $I(t)$ is current flowing in through the external bond attached to site 1. Since for Laplace transforms the $\lim_{t\to\infty}f(t)=\lim_{s\to 0}s\hat{f}(s)$, the resistance is

$$r(c_a,c_b)=\frac{(c_a-c_b)[x(0)-y(0)]}{c_a[1-x(0)]-c_b y(0)}. \tag{5.4}$$

But, the resistance $r(c_a,c_b)$ should not depend on c_a and c_b. In particular, the current should be independent of the sum c_a+c_b. Thus for current conservation in the static limit

$$\phi_1=1-x-y=0, \tag{5.5}$$

and hence

$$r=(1-2y)/y. \tag{5.6}$$

The constraint Eq. (5.5) is one of the desired invariants. Point (x,y) initially on the line $\phi_1=0$ corresponds to a Kirchhoff steady state, which iterates into another point (X,Y) on the same line. To see this, factor $\phi_1(X,Y)$ substituting from Eqs. (3.2):

$$1-X-Y=\frac{(1-x+y)(1-x-y)}{1-x}. \tag{5.7}$$

This factorization computes the preimage of the zero set of ϕ_1. Since $\phi_1(x,y)$ divides $\phi_1(X(x,y),Y(x,y))$, a point initially on $\phi_1(x,y)=0$ remains there.

The preimage of ϕ_1 contains also

$$\phi_2=1-x+y=0. \tag{5.8}$$

The line $\phi_2=0$ is invariant under any group admitted by the recursions, but not under the recursions themselves.

The pivotal Green function recursions Eq. (3.5) contain also the recursion for resistance of the chain as their static limit. For $\phi_1=0$ the Y recursion depends only on y, namely $Y=y/(2-y)$. Using Eq. (5.6) to change variables gives the expected resistor recursion $R=2r+1$. Hence current conservation in the steady state gives both an invariant line and the static scaling. In the anisotropic simplex example the corresponding result will appear less trivial.

The dynamical system $\mathcal{S}$ represented by recursion Eqs. (3.2) has a line of fixed points at $y=0$ and another invariant line $x=0$. One finds another invariant line,

$$\phi_3=1+x-y=0, \tag{5.9}$$

and in its preimage

$$\phi_4=1+x+y=0. \tag{5.10}$$

If $\mathcal{S}$ admits a group $\mathcal{G}$ then the lines comprising the zero sets of x, y, ϕ_1, ϕ_2, ϕ_3, and ϕ_4 must be trajectories of $\mathcal{G}$. The system $\mathcal{S}$ also has fixed points at ∞. $\mathcal{G}$ has fixed points wherever two trajectories intersect so (0,0), (1,0), (−1,0), (0,1), and (0,−1) are fixed by $\mathcal{G}$ in the finite plane. An efficient strategy is the following. Taking advantage of the connection between the Schrödinger problem and diffusion, transform from the wave mechanical Green functions (x,y) to Green functions (x_c,y_c) for diffusion in the completely closed chain. The recursions for (x_c,y_c) may be expected to simplify since material conservation is taken into account automatically. Using the general Eq. (2.8) with B as defined above, the coordinate change is

$$x_c=\frac{x-x^2+y^2}{(1-x+y)(1-x-y)},$$
$$y_c=\frac{y}{(1-x+y)(1-x-y)}, \tag{5.11}$$

in terms of which the recursions are

$$X_c=x_c-\frac{y_c^2}{1+2x_c},\quad Y_c=\frac{y_c^2}{1+2x_c}. \tag{5.12}$$

There are no isolated fixed points. The line y_c is fixed. The mass conservation line is now at ∞, and the inverse image of ∞ is the line $1+2x_c=0$. The lines $y_c=0$ and $1+2x_c=0$ must be trajectories of any group $\mathcal{G}$ admitted by the (x_c,y_c) recursions. These intersect at (−1/2,0). By continuing to take inverse images, one can construct an unlimited number of group trajectories. However, the only intersection in the finite plane of any pair of these trajectories is at (−1/2,0). This is a fixed point of any group that permutes solutions of the renormalization map for (x_c,y_c).

The next step is to translate coordinates to put the group fixed point at the origin. Taking $x_f=x_c+1/2$ and $y_f=y_c$,

$$X_f=\frac{2x_f{}^2-y_f{}^2}{2x_f},\quad Y_f=\frac{y_f{}^2}{2x_f}. \tag{5.13}$$

In this example the new recursions are homogeneous, so the group flow is radial. The recursions admit the group

$$\tilde{x}_f=e^{\alpha}x_f,\quad \tilde{y}_f=e^{\alpha}y_f, \tag{5.14}$$

with canonical variables $b=x_f/y_f$, $c=\ln y_f$. The decoupled recursions are finally

$$B=2b^2-1, \quad C=c-\ln(2b). \tag{5.15}$$

Notice that the selection of canonical coordinates is similar at this point to selecting dimensionless combinations of physical numbers. We seek dimensional or isobaric groups. The solution $b=f(\theta_o^{2^n})$, where $f(\cdot)$ is $\cos(\cdot)$ when $b^2<1$ and $\cosh(\cdot)$ for $b^2\geq 1$, is well known. The solution for c is a summation. Using the coordinate transformations from (a,b) back to (x,y) solves the original Scrödinger problem.

There are many ways to solve for Laplacian Green functions on a linear chain [32–34]. It is no surprise that an exact solution is possible. The point in this case is that the steps leading to the decoupling, i.e., all steps but the last, work equally well on a wider class of fractals. The result is that in each case the order of the recursions is reduced. Often this leads to a very complete analysis of the underlying lattice problem.

In summary, the steps are as follows: (1) transform the problem to a diffusion model with closed boundary conditions. (2) Obtain invariant manifolds of the group as inverse images of fixed sets of the recursions. The intersection of these will contain group fixed points. Typically the intersection is a single fixed point. (3) Move the fixed point to the center of coordinates. (4) Look for products of powers of the new coordinates to be invariant under the recursions. These form the b's of a set of canonical variables, i.e., the ones left fixed by the group flow. Each distinct choice of the invariant products gives another group. (5) A canonical c for each group, i.e., a coordinate along the flow is the logarithm of any convenient variable used to form the b products.

We illustrate by applying the method to each of the three remaining lattice examples introduced above.

The symmetrized Green functions for the Schrödinger equation on the anisotropic simplex Fig. 2 are defined in Eqs. (3.6) and renormalize according to Eqs. (3.7). These pertain to the case where the Hamiltonian H is just the adjacency matrix A. To obtain a diffusion model, let $z=s+2+k$ be the argument in the general resolvent $G(z)$ of Eq. (2.16). To close the system with respect to loss of material at the corner points requires a boundary transformation $B_{ij}=\delta_{i1}\delta_{j1}+\delta_{i2}\delta_{j2}+k\delta_{i3}\delta_{j3}$. The rescaled Green functions corresponding to (p,q,r,s) in the case of diffusion in the closed system are

$$\begin{aligned} p_c&=\frac{p-pr+2s^2}{1-p-r+pr-2s^2},\\ q_c&=\frac{q}{1-q},\\ r_c&=\frac{r-pr+2s^2}{1-p-r+pr-2s^2},\\ s_c&=\frac{s}{1-p-r+pr-2s^2}. \end{aligned} \tag{5.16}$$

The recursions become

$$\begin{aligned} P_c&=q_c+\frac{(p_c-q_c)(1+p_c+q_c)}{2+3p_c+q_c},\\ Q_c&=q_c+\frac{(1+2q_c)(p_c-q_c+2p_cr_c-2q_cr_c-4{s_c}^2)}{2+p_c+3q_c+4r_c+2p_cr_c+6q_cr_c-4{s_c}^2},\\ R_c&=r_c-\frac{4{s_c}^2}{2+3p_c+q_c},\\ S_c&=\frac{(p_c-q_c)s_c}{2+3p_c+q_c}. \end{aligned} \tag{5.17}$$

The two-dimensional fixed plane defied by setting both p_c-q_c and s_c to zero contains all finite fixed points of the dynamical system Eqs. (5.17). The inverse image of ∞ obtained by factoring the reciprocals of Eq. (5.17) intersects the fixed plane at the unique point $(-1/2,-1/2,-1/2,0)$. All inverse images one can construct contain this unique point in common. It is a candidate for a group fixed point. Thus we define new coordinates centered at the fixed point. In terms of these the recursions are

$$\begin{aligned} P_f&=\frac{p_f(p_f+3q_f)}{3p_f+q_f},\\ Q_f&=\frac{q_f(3p_fr_f+q_fr_f-6{s_f}^2)}{p_fr_f+3q_fr_f-2{s_f}^2},\\ R_f&=\frac{3p_fr_f+q_fr_f-4{s_f}^2}{3p_f+q_f},\\ S_f&=\frac{(p_f-q_f)s_f}{3p_f+q_f}. \end{aligned} \tag{5.18}$$

To find the possible isobaric groups, we apply dimensional analysis. Clearly q_f and p_f must have the same isobaric weight while s_f can have different weight. The weight of r_c is determined by the choices of weights for p_c and s_c. Thus the recursions admit the scale transformations

$$\tilde{p}_f=e^{\alpha}p_f, \quad \tilde{q}_f=e^{\alpha}q_f, \quad \tilde{r}_f=e^{2\beta-\alpha}r_f, \quad \tilde{s}_f=e^{\beta}s_f. \tag{5.19}$$

Equation (5.19) defines the product of operations of two commuting groups, one parametrized by α and the other by β. Thus it is possible to define two scale-independent products, a and b, and two other canonical variables by

$$a=\frac{{s_f}^2}{p_fr_f}, \quad b=\frac{p_f}{q_f}, \quad c=\ln\left(\frac{s_f}{r_f}\right), \quad d=\ln s_f. \tag{5.20}$$

The decoupled system is finally

$$A=\frac{a(1-b)^2}{(3+b)(1+3b-4ab)},$$

$$B=\frac{b(3+b)(3+b-2ab)}{(1+3b)(1+3b-6ab)},$$

$$C=c+\ln\left(\frac{b-1}{1+3b-4ab}\right), \tag{5.21}$$

$$D=d+\ln\left(\frac{b-1}{1+3b}\right).$$

Thus, because the recursions admit two groups, the order is reduced by two.

Equations (5.21) may not reduce further as they represent renormalization of the physical parameters s and k. The initial values for (a,b) of Eqs. (5.21) depend on k and on $z=s+2+k$. If we identify (a,b) after one iteration of Eq. (5.21) with initial conditions expressed in terms of new k and z values we generate recursions:

$$K=\frac{k(z+k-1)(z+k+1)}{(kz-k^2+1)(kz+k^2+1)},$$

$$Z=\frac{(z^2-k^2-3)(z+k)}{(kz+k^2+1)}-K. \tag{5.22}$$

The only z-independent fixed points are $k=0$ and $k=1$, which represent the linear chain and isotropic simplex limits. The line $2-z-k=0$ is invariant, meaning the static limit $s=0$, or the edge $E=2+k$ of the energy spectrum in the Schrödinger problem, corresponds from one generation to the next. Using this fact one can find the spectral dimension [35].

Renormalization equations (5.21) or (5.22) derive from consideration of diffusion. But any such recursions apply immediately to the quantum mechanical or vibrational models as well. The change of time dependence is straightforward and the chance of boundary conditions via any boundary matrix B that applies only to the connection points only generates a coordinate change, as we have seen, from one set of pivotal Green functions to another. Thus the solutions are transported back to the original Schrödinger problem using the coordinate changes defined in Eqs. (3.6), (5.16), and (5.20). The symmetry groups are conveniently found for the diffusion boundary conditions because of the connection to static scaling of electrical resistance. It is worth deriving the steady-state or static limit in the anisotropic lattice where it is less trivial than for the chain.

To obtain the resistance recursions for the anisotropic model, start with the diffusion problem for the closed system and then attach reservoirs at sites 1 and 2 with concentrations c_a and c_b, respectively. This requires a different change in boundary conditions $B'_{ij}=-\delta_{i1}\delta_{j1}-\delta_{i2}\delta_{j2}$ and introduces source terms $\delta_{i1}c_a/s+\delta_{i2}c_b/s$. If (x_b,y_b) are the Green functions after correcting the boundary, the electrical resistance r_{12} between corners 1 and 2 is

$$r_{12}=\frac{x_b(0)-y_b(0)}{1-x_b(0)} \tag{5.23}$$

and the current conserving constraint is

$$\phi_1=(1-p)(1-r)-2s^2=0. \tag{5.24}$$

From Eqs. (5.16) this verifies that the current conserving states are at ∞ in p_c, r_c, and s_c. Computing r_{13} requires reservoirs attached at sites 1 and 3, so yet another boundary change is required. If we subscript the Green functions with a in this case,

$$r_{13}=\frac{x_a(0)-kv_a(0)}{1-x_a(0)}. \tag{5.25}$$

The current constraint depends on k in this case even though k has been scaled out of the recursions. In terms of (p,q,r,s) the conservation condition is

$$\phi_2=k(1-r)-2(1-p)=0\,. \tag{5.26}$$

Another way to obtain the constraints is to attach reservoirs of concentration 1 to all the corner sites at once. This results in

$$\phi_3=1-p-s\sqrt{k}=0 \tag{5.27}$$

and

$$\phi_4=(1-r)\sqrt{k}-2s=0\,. \tag{5.28}$$

ϕ_1 is found on eliminating k between ϕ_3 and ϕ_4 while eliminating s gives ϕ_2. Equations (5.24) and (5.26) can be seen as requiring simultaneous conservation of charge (or mass) and anisotropy k. We need k to be a constant of the motion so that quantities on successive generations continue to refer to a model with the same k value. When we try to visualize the recursions in terms of renormalizing z and k in Eqs. (5.22), their coupled evolution becomes inconvenient.

Modulo $\phi_3=0$ and $\phi_4=0$ the recursions for p and q decouple in Eqs. (3.7):

$$P=q+\frac{(1-q)(1-pq)}{2-p(1+q)},$$

$$Q=q+\frac{(1-q)(1+q)[(1-p)(1-q)-k(p+q)]}{(2+q)(1-p)(1-q)-k(2-pq-q^2)}, \tag{5.29}$$

and the resistances in terms of Green functions at $s=0$ simplify to

$$r_{12}=\frac{q}{1-q},\quad r_{13}=\frac{k(p+q-2pq)-2(1-p)}{2k(1-p)(1-q)}. \tag{5.30}$$

Thus, changing coordinates from (p,q) to (r_{12},r_{13}) one obtains the resistor recursions

$$R_{12}=r_{12}+\frac{(1+2r_{12})(1-2kr_{12}+2kr_{13})}{1+2k(1+r_{12}+r_{13})},$$

$$R_{13}=r_{13}+\frac{(1+2r_{12})[1+k(1+2r_{13})]}{1+2k(1+r_{12}+r_{13})}. \tag{5.31}$$

The latter equations can be verified by circuit theory. The derivation shows how mass conservation locates an invariant curve. It can be found by enforcing current conservation in one of several ways. One is to require that resistance not depend on reservoir concentrations. Another is to attach reservoirs of equal concentration to the model with open boundary conditions at all external sites (Schrödinger or vibration problem) and require the same concentration at each corner in the long-time limit. The most efficient way is to transform to closed (diffusion) boundary conditions. A group trajectory is then obtained as the inverse image of ∞. By clearing an invariant curve of the dynamical system from the finite plane, the latter procedure usually simplifies the group flow. Often only a line or plane of fixed points of the system remains, and the set where the inverse image of infinity intersects this will locate a fixed point of the group.

In the anisotropic example, the symmetry group is Abelian. An example of a non-Abelian group is the one admitted by the recursion Eqs. (3.9) belonging to the lattice of Fig. 3. In such a case the question of solvability arises. When $\mathcal{S}$ admits an r-parameter group, the most obvious strategy is to apply the reduction outlined in Sec. IV using a one-parameter subgroup, thus reducing the order by one. But then one must ask whether the new system still admits a group of $r-1$ parameters. The criterion for a full reduction by r variables is that the original group be solvable [17]. The reduction of order via a solvable group is discussed by Maeda [18].

For the system of equations (3.9) the mass-conservation hyperplane does not appear at infinity. If one attaches reservoirs of equal concentration c_0 to each connection site, then the saturation condition is seen to imply a zero of either $\phi_1 = x+y+t+u+2v-1$ or $\phi_2 = p+m-1$. The large-size limit is a zero either of each of u, v, and t simultaneously or else of r, s, and $1/m$ simultaneously. Taking inverse images we can construct enough group invariant sets to determine three one-parameter groups, each of which commutes with the recursions, namely, the groups generated by

$$\mathcal{L}_1 = m\frac{\partial}{\partial m} - \left(\frac{1}{4}-p\right)\frac{\partial}{\partial p} + q\frac{\partial}{\partial q} + r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s},$$

$$\mathcal{L}_2 = \frac{\partial}{\partial q}, \tag{5.32}$$

$$\mathcal{L}_3 = m\frac{\partial}{\partial m} - \left(\frac{1}{4}-p\right)\frac{\partial}{\partial p} + \left(\frac{m}{4}+\frac{r}{2}\right)\frac{\partial}{\partial r} - \left(\frac{m}{4}-\frac{r}{2}\right)\frac{\partial}{\partial s}.$$

When exponentiated, the infinitesimal generators Eqs. (5.32) reproduce the action of the groups. Together, they generate a solvable three-parameter group with Lie algebra characterized by the single nonvanishing commutator $[\mathcal{L}_2, \mathcal{L}_1] = \mathcal{L}_2$. With the generators one can get canonical variables, as described by Stephani [17,16],

$$a = \ln\left[m\left(1-\frac{2r}{m}\right)^2\right],$$

$$b = \ln m, \quad c = \ln(m-2r+2s), \tag{5.33}$$

$$d = q, \quad f = \frac{1-4p}{m}.$$

The reduced recursions are

$$A = a + \ln\left(\frac{1}{f+1}\right) + \ln\left(\frac{f-2}{f+2}\right),$$

$$B = b + \ln\left(\frac{1}{f+1}\right) - \ln\left(\frac{f-2}{f+2}\right),$$

$$C = a + \ln\left(\frac{1}{f+1}\right), \tag{5.34}$$

$$D = d + \frac{e^a}{f+1},$$

$$F = (f-3)f.$$

Thus a, b, and c decouple, f satisfies an independent recursion in one variable, and d is a sum involving the orbits of f and a. Recursion f is related to eigenvalue recursion one can find from decimation [8,9].

Finally, consider diffusion on the Vicsek lattice [23] of Fig. 4 with closed boundary conditions. The diffusion operator is not the adjacency matrix in this case and the transformation B connecting the open and closed diffusion models does not act only at the connection points, so it will not be an easy matter to map the solution from the closed case addressed here to the open one more closely related to vibrational models. Transverse vibrations on a Vicsek lattice with general boundary conditions have been studied recently [25].

The renormalization of (x,y) is given by Eqs. (3.10). Fixed points in the finite plane comprise only the line $y=0$. The inverse image of ∞ is the union of

$$\phi_1 = 1+2x-y=0\,, \quad \phi_2 = 1+2x+3y=0\,. \tag{5.35}$$

These three lines must be group trajectories. They intersect at $(-1/2,0)$. Moving the origin to this point by $x_f = x+1/2$, $y_f = y$ one has the recursions

$$X_f = \frac{4x_f^3 + 4x_f^2 y_f - 5x_f y_f^2 - 2y_f^3}{(2x_f - y_f)(2x_f + 3y_f)},$$

$$Y_f = \frac{y_f^3}{(2x_f - y_f)(2x_f + 3y_f)}. \tag{5.36}$$

The equations are homogeneous, i.e., they admit uniform dilation, thus canonical variables are $b = x_f/y_f$ and $c = \ln y_f$. The decoupled recursions are

$$B = 4b^3 + 4b^2 - 5b - 2\,, \quad C = c - \ln[(2b-1)(2b+3)]. \tag{5.37}$$

The b recursion relates the Cantor-set portion of the eigenvalue spectrum (see Ref. [8]). To see this, use the initial condition $x=y=1/s$. The eigenvalues with weight at the distal site of the lattice are the singularities of the Green functions. This portion of the spectrum consists of singularities in (x,y) at generation $n+1$ caused by singularities of (x,y) at generation n. Using the initial values and the definition of the coordinate b gives an s recursion

$$S=s(3+s)(5+s), \tag{5.38}$$

meaning that if S belongs to the Cantor set, the three roots $s_p(S)$, $p\in\{1,2,3\}$ also belong to the Cantor set. The spectral minimum is $s=-4-\sqrt{2}$ and the spectral dimension obtained from $S\sim 15s$ at small s is $d_s=2\ln 5/\ln 15=1.189$, which is a standard result [25]. Notice that although a general solution of the b recursion in Eq. (5.37) is not known, one can still find an infinite set of solutions as explicit functions of n [29]. Suppose, for example, s is such that the initial value $b,0$ belongs to a 3-cycle with members $\{b_1=b_0,b_2,b_3\}$:

$$\begin{aligned} C_{3n}&=-n\left(\sum_{m=1}^{3}\ln[(2b_m-1)(2b_m+3)]\right),\\ C_{3n+1}&=C_{3n}-\ln[(2b_1-1)(2b_1+3)],\\ C_{3n+2}&=C_{3n+1}-\ln[(2b_2-1)(2b_2+3)]. \end{aligned} \tag{5.39}$$

As special as these solutions seem, they have been shown to correspond to a set of transmission resonances in the electron propagation problem [29,31].

VI. SUMMARY

We have shown how to find Lie groups that commute with the real-space renormalization of dynamical problems on regular fractals. Each such group reduces the number of variables in the recursions. Thus a partial and sometimes a total decoupling can be obtained. The procedure has been demonstrated with four examples: diffusion (or vibrations or electron propagation) on the line, an anisotropic 3-simplex, a fourfold coordinated Sierpiński lattice, and a Vicsek lattice. It has also been successful for a dozen or so other lattice types, many of which are actually multiparameter families of lattices [29].

One must admit that the recursion equations dealt with in the preceding sections come from a class of difference equations with rather special properties. One reason for choosing regular fractal problems is that they renormalize exactly. The difference equations correspond, however abstractly, to physical problems that carry with them some deep symmetries. To solve approximate renormalization equations exactly by finding groups they admit would be of less interest, certainly. We suspect such a project would also be less likely to succeed. However, the uniform dilation symmetry found in each example above, as well as most others, is a manifestation of the fact that scaling each resistor in a network by λ scales the entire network resistance by λ. It is related clearly to current conservation. It does not go away when symmetry of the lattice is reduced.

The techniques introduced above do not appear to work well for statistical dynamical problems on hierarchical lattices, even though the latter renormalize exactly and ought to admit groups relating to inflation. Such statistical models include spin Hamiltonians, percolation, self-avoiding walks, etc. The recursions are often polynomial. There is no mass current to conserve, the inverse image of ∞ is ∞.

However, for dynamical systems where a symmetry group exists, the strategy of Sec. V may be of use for finding it. It is global as opposed to the method proposed by Quispel and Sahadevan [19] in which one attempts to construct a series solution of the group generator about a fixed point of the difference equations. With luck one can hope to sum the series and then integrate the resulting generator to obtain group coordinates. Both methods require luck, and they are to a large extent complementary to one another.

The steps to follow for finding a group that commutes with renormalization of a Laplacian based difference scheme are summarized already in Sec. V. For more general dynamical systems the part one might hope to generalize consists of the following. First find fixed or invariant sets of the difference equations. These must also be invariant sets of the group flow. When the recursions consist of rational functions, both invariant sets and their inverse images can be found by factorization, as we have illustrated. The inverse image must also be invariant under the group. With enough examples of group invariant sets, one can take intersections to find trajectories and fixed points. These are apt to be of value for constructing a group. Once found, a continuous symmetry group reduces the order of the difference equations as discussed in Sec. IV.

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